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THE NONLINEAR VISCOELASTIC COSSERAT MODEL FOR THE WAVE PROPAGATION WITH GENERATION OF SUBHARMONICS

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Abstract—A Cosserat model is developed in nonlinear form to describe the appearance of lower frequencies in seismic waves spectra. As in the Ericksen theory of liquid crystals, elastic and viscous rheological elements are combined to get sensible results. The translational degrees of freedom are described by linear elasticity but the rotational kinematically independent motion is governed by the nonlinear elastic potential and linear viscosity.

The dynamics of linear perturbations show that the longitudinal translations and rotations both decouple completely from other motions. These rotations represent local oscillations due to the existence of the potential and they do not propagate. However, the linear transverse translations and microrotations are coupled. It is shown that they describe seismic and acoustic waves.

In the nonlinear case these motions demonstrate some new interesting phenomena. For instance, if the propagation of the harmonic elastic wave is considered the microrotations behave like nonlinear oscillators excited by the external harmonic force. Thus the system produces the same effects as those obtained recently for the Duffing type equation with the help of the theory of attractors. This means that the initial harmonic wave generates secondary waves with lower frequencies. These secondary frequencies are usually commensurable with the initial ones. Numerical results show that the phenomenon still takes place if the initial wave consists of continuously distributed harmonics.

The generation of lower frequencies in granular media has been observed and reported but the theoretical explanation was lacking since it was thought conventionally that weak nonlinearities were able to produce higher frequencies only.

1. INTRODUCTION

The Cosserat theory (Cosserat and Cosserat, 1909; Ericksen and Trusdell, 1958; Trusdell, 1966) of kinematically independent dynamics of microstructure leads essentially to dynamics of two interpenetrating continua with interaction of a moment of momentum type. This model is close in spirit to the Frenkel–Biot poroelasticity (Biot, 1956; Frenkel, 1944) where the difference in translational velocities of solid and fluid particles is a basic ingredient. This mathematical construction is extremely useful for the dynamic description of naturally fragmented materials such as rock and soil. It serves to represent properties of real seismic waves, of ground noise, etc., which cannot be treated using conventional elasticity.

The significance of a continuum with microstructure in geomechanics must be estimated in comparison with other possible models. However a model with microstructure can be utilised in a wider range because of a multiscale character (in time and space) of geological processes. The key point in applications is to combine properly the rheological laws at different scales. The Ericksen–Leslie model of a liquid crystal (Ericksen, 1967; Leslie, 1968) can be used as an example of a proper approach.

We should like to stress that nonlinear version of the theory explaining many natural phenomena (Nikolaevskiy, 1996; Nikolaev and Galkin, 1987) reduce to equations which are well known in modern mathematical physics.

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2. BASIC EQUATIONS

The propagation of the waves in media with microstructure possesses some peculiar features due to the interaction of the translational and rotational degrees of freedom. In the present work we investigate the wave processes when the macroscopic motion is purely elastic and the microscopic motion is dissipative.

Previously models were considered in which the macroscopic flow was viscous and the dynamics of the microrotations were described by equations of the nonlinear elasticity type (Dinariev and Nikolaevskii, 1995a, b).

Here we consider the medium governed by the following sets of equations, the mass balance:

$$\frac{d\rho}{\rho dt}+\rho v_{i,i}=0, \tag{1}$$

the momentum balance:

$$\rho\frac{dv_i}{dt}=p_{ij,j}+f_i, \tag{2}$$

the moment of momentum balance:

$$\frac{d}{dt}(\varepsilon_{ijk}x_j\rho v_k+J\Omega_i)+(\varepsilon_{ijk}x_j\rho v_k+J\Omega_i)v_{l,l}=(\varepsilon_{ikl}x_k p_{lj}+\pi_{ij})_{,j}+\varepsilon_{ijk}x_jf_k+m_i, \tag{3}$$

and the energy balance:

$$\frac{d}{dt}(K+\rho U)+(K+\rho U)v_{l,l}=(v_ip_{ij})_{,j}+(\Omega_i\pi_{ij})_{,j}+f_iv_i+m_i\Omega_i+q_{i,i}+\varepsilon, \tag{4}$$

where the conventional summation is used for repeated indices.

We use an inertial frame of reference with time t and coordinates x_i, and the notation: ρ is the mass density, v_i the velocity, p_{ij} the total stress tensor, f_i the body force, J the scalar (for simplicity) moment of inertia (per unit volume), which is a tensor in a general case, Ω_i the total angular velocity, π_{ij} the couple stress tensor, m_i the moment of momentum, $K=(\varepsilon v_iv_i+J\Omega_i\Omega_i)/2$ the kinetic energy, U the internal energy (per unit mass), q_i the heat flux, ε the heat production. The material time derivative is

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}$$

and ε_{ijk} are components of the alternator tensor.

Introduce the displacement vector $u_i=u_i\,(t,x_j)$ and the total rotation vector $\varphi_i=\varphi_i\,(t,x_j)$. Then the velocities can be expressed as the derivatives

$$v_i=\frac{du_i}{dt},\quad \Omega_i=\frac{d\varphi_i}{dt}.$$

The system of equations (1)–(4) serves to determine the following variables:

$$\rho=\rho(t,x_j),\quad u_i=u_i(t,x_j),\quad \varphi_i=\varphi_i(t,x_j),\quad T=T(t,x_j)$$

where T is the temperature. In order to have complete formulation of the problem it is necessary to add to the constitutive relations, which determine the quantities p_{ij}, π_{ij}, q_i, m_i,

U. We shall formulate the constitutive relations in accordance with irreversible thermodynamics (de Groot and Masur, 1962). Assuming infinitesimal displacement, the approximations are, for velocities

$$v_i = \partial_t u_i, \quad \Omega_i = \partial_t \varphi_i,$$

for the strain tensor

$$\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$$

and for the vector of relative rotation

$$\varphi_i^* = \varphi_i + (1/2)\varepsilon_{ijk} u_{j,k}.$$

Suppose that the internal energy U depends on $T, \varepsilon_{ij}, \varphi_i^*$ only, then from (1) the mass density ρ changes are related to the strain tensor by

$$\rho = g\rho_0, \quad g = \det(\delta_{ij} - \varepsilon_{ij}),$$

where ρ_0 is the initial density and δ_{ij} are components of the unit tensor.

The Gibbs equation corresponding to our choice of the thermodynamic parameters of the system is

$$T\,\mathrm{d}s = \mathrm{d}U - \rho^{-1}(\sigma_{ij}\,\mathrm{d}\varepsilon_{ij} + v_i\,\mathrm{d}\varphi_i^*). \tag{5}$$

Here s is the entropy (per unit mass), σ_{ij} is the elastic stress tensor and v_i is the elastic moment of momentum generated by the microrotations. The following expressions are consequences of eqn (5):

$$\sigma_{ij} = \left(\frac{\partial U}{\partial \varepsilon_{ij}}\right)_s, \quad v_i = \left(\frac{\partial U}{\partial \varphi_i^*}\right)_s. \tag{6}$$

The entropy time derivative is given by eqns (2)–(6)

$$\rho T \frac{\mathrm{d}s}{\mathrm{d}t} = (p_{ij} - \sigma_{ij})v_{i,j} + \pi_{i,j}\Omega_{i,j} + \varepsilon_{ijk}\Omega_i p_{jk} + \varepsilon - q_{i,i} - v_i \Omega_i^*, \tag{7}$$

where $\Omega_i^* = \partial_t \varphi_i^*$ is the relative angular velocity. The viscous stress tensor

$$\tau_{ij} = p_{ij} - \sigma_{ij}$$

is defined as the difference between total and elastic stresses. In general it is composed of the symmetric $\tau_{ij}^s = \tau_{(ij)}$ and the antisymmetric part $\tau_{ij}^a = \tau_{[ij]}$.

The entropy production (Truesdell, 1972) is given by

$$\Pi = \rho \frac{\mathrm{d}s}{\mathrm{d}t} - T^{-1}\varepsilon + (q_i T^{-1})_{,i},$$

which can be expressed with the help of (7)

$$\Pi = T^{-1}(\tau_{ij}^s v_{i,j} + \tau_{ij}^a(v_{[i,j]} + \varepsilon_{ijk}\Omega_k) + \pi_{ij}\Omega_{i,j} - v_i\Omega_i^*) + q_i(T^{-1})_{,i}. \tag{8}$$

According to the second law of thermodynamics, this quantity must not be negative (Trusdell, 1972):

$$\Pi \geqslant 0. \tag{9}$$

We consider only dynamical processes which have zero dissipation in deformation processes governed solely by translational motions, that is when $\tau^s_{ij} = 0$. Then introducing the notation

$$\xi_i = \tau_{jk}\varepsilon_{ijk} - v_i, \quad \kappa_{,i} = -T^{-1}T_{,i},$$
$$\psi = \Omega_{i,i}, \quad \psi_{ij} = \Omega_{(ij)} - (1/3)\delta_{ij}\psi,$$
$$\gamma = \pi_{i,i}, \quad \gamma_{ij} = \pi_{ij} - (1/3)\delta_{ij}\gamma,$$

we reduce expression (8) to the form

$$\Pi = T^{-1}(\xi_k\Omega^*_k + \gamma\psi + \gamma_{ij}\psi_{ij} + q_i\kappa_i).$$

According to the Onsager principle (de Groot and Masur, 1962) this expression and the rotational invariance (The Pierre Curie principle) yields the following linear kinetic laws for dissipative fluxes:

$$\gamma = a_1\psi, \quad \gamma_{ij} = a_2\psi_{ij},$$
$$\xi_i = \beta_1\Omega^*_i, \quad q_i = \beta_2\kappa_i,$$

where we introduced the kinetic coefficients, a_1, a_2, β_α. According to (9) the inequalities

$$a_1 \geqslant 0, \quad a_2 \geqslant 0, \quad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0,$$

have to be valid.

In order to make the next step it is necessary to specify the functional form of the internal energy U.

3. LINEAR DYNAMICS CASE

Consider small dynamical perturbations of the initial state

$$\rho = \rho_0, \quad T = T_0, \quad u_i = 0, \quad \varphi_i = 0. \tag{10}$$

In the lowest second-order approximation for displacements the internal energy is

$$U = U_0(T) + (2\rho_0)^{-1}(\lambda_1(\varepsilon_{ii})^2 + \lambda_2\varepsilon_{ij}\varepsilon_{ij}) + \rho_0^{-1}W, \tag{11}$$

where the elastic potential W for the microrotations is

$$W = W(\varphi^*_i) = (1/2)\zeta\varphi^*_i\varphi^*_i, \tag{12}$$

and λ_1, λ_2, ζ are constants. The initial state (10) is assumed to be stable, which implies the inequalities

$$\lambda_1, \lambda_2, \zeta > 0.$$

The corresponding elastic stress components are determined by (6), (11), (12) are the linear approximations

$$\sigma_{ij} = \lambda_1\varepsilon_{kk}\delta_{ij} + \lambda_2\varepsilon_{ij}, \quad v_i = \zeta\varphi^*_i. \tag{13}$$

Denote the double Fourier transform of any field $f = f(t, x_i)$ as

$$f_F(\omega, k_i) = \int \exp(-i\omega t - ik_i x_i) f(t, x_i)\, \mathrm{d}t\, \mathrm{d}x_j,$$

where ω and k_i are, respectively, frequency and wave numbers, and introduce the notation

$$\vartheta = T - T_0, \quad C_v = \frac{\mathrm{d}U_0(T)}{\mathrm{d}T}, \quad k^2 = k_i k_i.$$

Equations (2)–(4) and (13) after linearization yield the system of equations for the Fourier transforms:

$$\begin{aligned} 0 &= i\omega\rho_0 v_{iF} - ik_j p_{ijF} = A^1_{ij} u_{jF} + B^1_{ij}\varphi_{jF}, \\ 0 &= i\omega J\Omega_{iF} + \varepsilon_{ijk} p_{jkF} - ik_j \pi_{ijF} - M_{iF} \\ &\doteq A^2_{ij} u_{jF} + B^2_{ij}\varphi_{jF} + C^2_i \vartheta_F, \\ 0 &= i\omega\rho_0 C_V \theta_F + ik_i q_{iF} = B^3_j \varphi_{jF} + C^3 \vartheta_F. \end{aligned} \tag{14}$$

The coefficients of this system are tensors of the second-, first- and zero-order. They can be represented in a form which explicitly takes into account $SO(3)$-invariance by decomposing them into the sum of fixed linear independent tensors:

$$\begin{aligned} A^\alpha_{ij} &= A^{\alpha 1}\delta_{ij} + A^{\alpha 2} k_i k_j + A^{\alpha 3}\varepsilon_{ijk} ik_k, \\ B^\alpha_{ij} &= B^{\alpha 1}\delta_{ij} + B^{\alpha 2} k_i k_j + B^{\alpha 3}\varepsilon_{ijk} ik_k, \\ C^2_i &= ik_i C^2, \quad B^3_i = ik_i B^3, \quad \alpha = 1, 2. \end{aligned}$$

The scalar coefficients $A^{\alpha\beta}$, $B^{\alpha\beta}$, C^2, B^3, C^3 can be calculated directly from (2)–(4):

$$\begin{aligned} A^{11} &= \frac{1}{2}\lambda_2 k^2 - \omega^2\rho_0 + \frac{1}{2}\left(i\omega\beta_1 + \frac{1}{2}\zeta\right)k^2, \\ A^{12} &= \lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{4}(i\omega\beta_1 + \zeta), \\ B^{13} &= \frac{1}{2}(i\omega\beta_1 + \zeta), \quad A^{23} = \frac{1}{2}(i\omega\beta_1 + \zeta), \\ B^{21} &= i\omega\beta_1 + \zeta - J\omega^2 + \frac{1}{2}a_2 i\omega k^2, \\ B^{22} &= \frac{1}{3}\left(a_1 + \frac{1}{2}a_2\right)i\omega, \quad C^2 = 0, \\ B^3 &= 0, \quad C^3 = k^2\beta_2 T_0^{-1} + i\omega\rho_0 C_V. \end{aligned}$$

To investigate different modes we calculate the determinant of (14)

$$\Delta = P_1 P_2 P_3^2,$$

where

$$\begin{aligned} P_1 &= A^{11} + k^2 A^{12} = (\lambda_1 + \lambda_2)k^2 - \omega^2\rho_0, \\ P_2 &= B^{21}C^3 + (B^{22}C^3 + B^3C^2)k^2, \end{aligned}$$

$$P_3 = A^{11}B^{21} - k^2 A^{23} B^{13}.$$

The general dispersion relation $\Delta = 0$ for the wave velocity can be decomposed into the three equations

$$P_\alpha = 0, \quad \alpha = 1, 2, 3.$$

The first equation $P_1 = 0$ evidently describes the longitudinal-translation waves which do not couple with other motions. The equations $P_2 = 0$ and $P_3 = 0$ describe the longitudinal rotational waves and the coupled transverse translational-rotational waves correspondingly.

In order to demonstrate this, first consider the dispersion relations in case of zero dissipation ($\Pi \equiv 0$):

$$P_2 = i\omega\rho_0 C_V(\zeta - J\omega^2) = 0. \tag{15}$$

This equation has the solution

$$\omega = \pm\omega_1, \quad \omega_1 = (\zeta/J)^{1/2} \tag{16}$$

which corresponds to the rotational oscillations. Further, we have

$$P_3 = \left(\frac{1}{2}\left(\lambda_2 + \frac{1}{2}\zeta\right)k^2 - \omega^2\rho_0\right)(\zeta - J\omega^2) - \frac{1}{4}\zeta^2 k^2 = 0. \tag{17}$$

Assuming a solution in the form

$$k = \alpha_1 \omega, \tag{18}$$

eqn (17) becomes

$$\alpha_1^2\left(\left(\lambda_2 + \frac{1}{2}\zeta\right)J\omega^2 - \zeta\lambda_2\right) = 2\rho_0(J\omega^2 - \zeta).$$

It follows that there are no wave solutions in the frequency range

$$\omega_2 \leqslant |\omega| \leqslant \omega_1, \quad \omega_2 = \omega_1(1 + (2\lambda_2)^{-1}\zeta)^{-1/2},$$

but solutions exist in the range

$$\omega_1 < |\omega| \quad \text{and} \quad |\omega| < \omega_2.$$

The former are called the "acoustic" (high frequency) waves. They correspond to microstructure (fragment scale) oscillations. The latter are called the seismic waves and can be observed by conventional tools in seismology.

Dissipation affects the solutions of eqns (15) and (17). For simplicity assume that there is only one nonzero viscosity

$$\beta = \beta_1 > 0, \tag{19}$$

and the dissipative terms in the dispersion relations are relatively small. Then the corrections of the first-order to (16), (18) are the following:

$$\omega = \pm\omega_1 + i\beta(2J)^{-1},$$

$$k = \alpha_1\omega + \alpha_2\beta, \quad \alpha_2 = -i\rho_0 J^2 \alpha_1^{-1}\left(\left(\lambda_2 + \frac{1}{2}\zeta\right)J\omega^2 - \zeta\lambda_2\right)^{-2}\omega^6.$$

So the friction coefficient β, which determines the viscous interaction of translational and rotational motions, accounts for damping of fragment (microstructure) oscillations and attenuation of the transverse waves.

4. NONLINEAR SPECTRA TRANSFORMATION

Now we turn our attention to possible nonlinear effects. Let the processes be isothermal and depend only on one space coordinate $x = x_1$, and let the variables be represented in the following functional form

$$u_i = \delta_{i2}u(t,x), \quad \varphi_i^* = \delta_{i3}\varphi(t,x).$$

Assume nonlinearity of the model by taking into account the higher-order terms in eqn (12), then

$$W(\varphi) = \frac{1}{2}\zeta\varphi^2 + \frac{1}{3}\zeta_1\varphi^3 + \frac{1}{4}\zeta_2\varphi^4. \tag{20}$$

The stability condition of the initial rest state (10) requires that

$$\zeta > 0, \quad \zeta_2 > 0.$$

Again we consider the model with only one viscous coefficient (19). Equations (2) and (3) give us coupled equations for $u(t,x)$, $\varphi(t,x)$ as the unknown variables:

$$\rho_0\,\partial_t^2 u - \frac{1}{2}\lambda_2\,\partial_x^2 u = -\frac{1}{2}\partial_x\left(\beta\,\partial_t\varphi + \frac{\mathrm{d}W}{\mathrm{d}\varphi}\right), \tag{21}$$

$$J\,\partial_t^2\varphi + \beta\,\partial_t\varphi + \frac{\mathrm{d}W}{\mathrm{d}\varphi} = -\frac{1}{2}J\,\partial_t^2\,\partial_x u, \tag{22}$$

where

$$\frac{\mathrm{d}W}{\mathrm{d}\varphi} = \zeta\varphi + \zeta_1\varphi^2 + \zeta_2\varphi^3.$$

Solutions of (21) and (22) can be found by the following procedure:

(a) construct a solution of (21) with zero right-hand side,
(b) substitute this into the right-hand side of (22) and find the solution for φ,
(c) substitute the latter into the right-hand side of (21) and find the correction to u, and so on.

This procedure is essentially one particular way to implement the nonlinear perturbation technique. It produces the convergent sequence, if the right-hand side in (21) is sufficiently small in comparison with every term in the left-hand side.

Let us consider what happens when at the first step we have a harmonic wave

$$u = -a_0 \cos(\omega t - kx), \quad \omega = kV, \quad V^2 = 2\rho_0/\lambda_2. \tag{23}$$

Substituting (23) and (22), we obtain the Duffing type equation for the damped nonlinear oscillator with harmonic external force. After some transformation it can be reduced to the canonical form

$$\begin{gathered}
\partial_\eta^2 \Phi + \lambda \partial_\eta \Phi + g_0 \Phi + g_1 \Phi^2 + \Phi^3 = A_0 \sin(2\pi(\eta - \eta_0)), \\
\varphi(t,x) = \zeta_2^{-1/2} J^{1/2} (2\pi)^{-1} \omega \Phi(\eta, x), \eta = (2\pi)^{-1} \omega t, \quad \eta_0 = (2\pi)^{-1} kx, \\
\lambda = 2\pi\beta\omega^{-1} J^{-1}, \quad g_0 = 2\zeta J^{-1/2} \omega^{-1} \zeta_2^{-1/2}, \\
g_1 = 2\pi\zeta_1 J^{-1/2} \omega^{-1} \zeta_2^{-1/2}, \quad A_0 = 2\pi^2 k a_0 \omega \zeta_2^{1/2} J^{-1/2}.
\end{gathered} \tag{24}$$

Equation (24) has been investigated in the context of the theory of attractors (Feigenbaum, 1980), which determines the general properties of solutions. Any solution can be determined by the point in the phase space Ω, $\partial_\eta \Omega$ (for example by the corresponding values at $\eta = 0$). The dynamics of (24) is adequately represented by the Poincaré map in the phase space

$$M: \quad (\Phi(0,x), \partial_\eta \Phi(0,x)) \to (\Phi(1,x), \partial_\eta \Phi(1,x)),$$

which transforms the phase point into the point given by the solution at the end of the period. This map is diminishing any phase volume by the factor $e^{-\lambda}$. Therefore, the stable solutions (attractors) are represented in the phase space by the sets of zero phase volume. A stable solution can consist of a finite number of phase points (so-called simple attractors) or of infinite number of points (so-called strange attractors). In the first case the solution is periodic with an integer period n. In the second case the stable solution is stochastic-like.

In both cases the response of the medium with microstructure to the initial wave (23) can contain lower frequencies. In the case of the simple attractor the response can be decomposed into the discrete set of harmonic oscillations with the spectrum $(m\omega/n)$ where m is an integer. This is a new phenomenon. Conventionally the nonlinear terms generate higher frequencies. Experimentally the appearance of lower frequencies has been observed (Vilchinskaya and Nikolaevskii, 1984; Guschin *et al.*, 1994; Bubnov *et al.*, 1994) in the field and in the laboratory, but with great disbelief due to absence of any theoretical background.

Let us discuss the possible values of n. According to Feigenbaum (1980) if to change the parameters a_0, ω of the initial wave (23) one can encounter the period doubling bifurcation sequence in the response of the medium with microstructure. However, a finite readjustment of the solution (not a mere bifurcation) can happen with any change of n.

Consider an attractor solution $\Omega = \Omega_0(\eta - \eta_0)$ of (24) and substitute it into (21). Then the correction to the initial wave (23) is described by

$$\rho_0 \partial_t^2 \Delta u - \frac{1}{2} \lambda_2 \partial_x^2 \Delta u = -\frac{1}{2} \partial_x \left(\beta \partial_t \varphi + \frac{\mathrm{d}W}{\mathrm{d}\varphi} \right) = F(t - V^{-1} x)$$

which can be solved explicitly:

$$\Delta u = \lambda_2^{-1} V x F_1 (t - V^{-1} x), \quad F_1 = \tfrac{1}{2} J \partial_t (\varphi + \tfrac{1}{2} \partial_x u).$$

Thus, the translational wave can really contain lower frequencies due to microrotations.

Now seismic signals are usually composed of harmonics distributed continuously over some frequency range. Let $\Delta\omega$ be the typical width of this distribution, then the quantity

$$\tau_*(\Delta\omega)^{-1}$$

characterizes the duration of the signal. The existence of an attractor can still be exhibited by the system until τ_* will be negligible in comparison with the time $(2J\beta^{-1})$ of the rotational oscillations.

Let the coefficients of function (20) be such that (20) has the form

$$W = W_0\varphi^2(\varphi-\alpha)^2.$$

The initial signal is characterized by the strain $\varepsilon = \varepsilon(t)$ which is distributed over a frequency range:

$$\varepsilon = \frac{1}{2}\partial_x u = 2\varepsilon_0\varepsilon_1^{-1}\int h(f)\sin(2\pi ft)\,\mathrm{d}f,$$

where

$$h(f) = \exp\left[-(\ln(f/f_0)/\ln(1+(\xi/f_0)))^2\right],\quad \varepsilon_1 = \int h(f)\mathrm{d}f,$$

and, say $f_0 = 2000$ Hz. [This frequency is chosen for comparison with the laboratory experiments (Guschin *et al.* 1994; Bubnov *et al.* 1994)]. The results of the computation are represented by the spectral density function, where the spectral density of any function $g = g(t)$ is defined by the expression

$$S(f) = \left|\int \exp(-i2\pi ft)g(t)\,\mathrm{d}t\right|.$$

In Fig. 1 the spectral density of the initial signal is normalized at the maximum value. The frequency distribution width ξ is equal to 150 Hz.

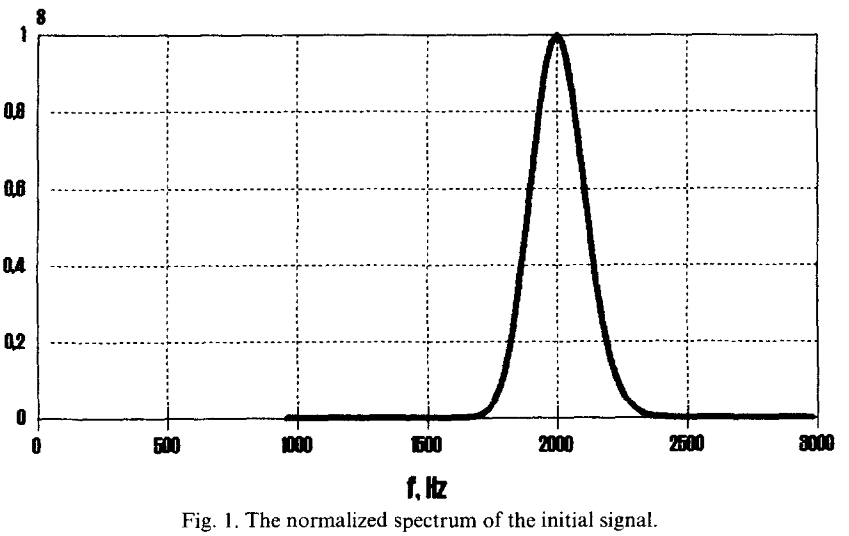

Fig. 1. The normalized spectrum of the initial signal.

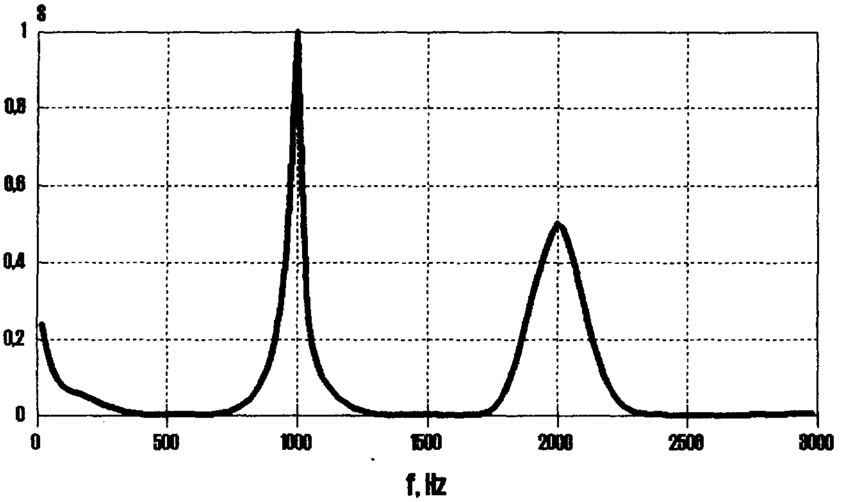

Fig. 2. The period doubling in the response spectrum.

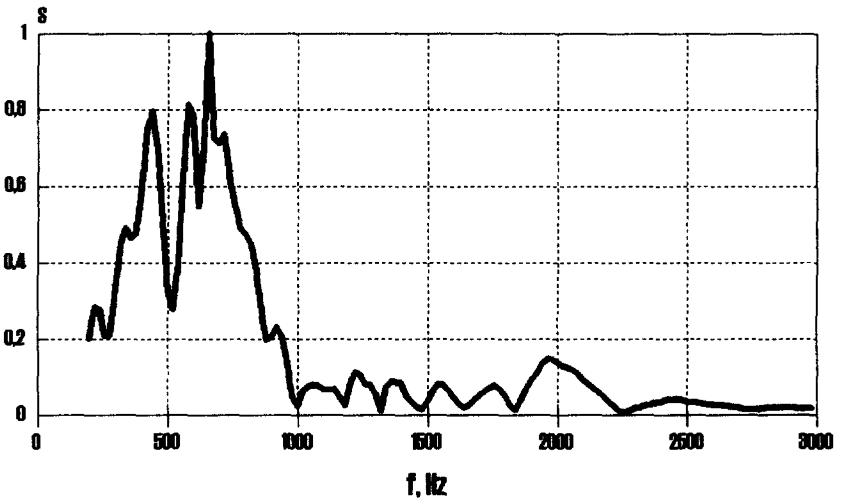

Fig. 3. The response spectrum. The broad peak looks like period tripling.

In Figs 2–4 the spectral density for the rotational response $\varphi(t)$ is normalized also at the maximum value. Parameters for these numerical experiments are chosen also in accordance with (Guschin *et al.*, 1994; Bubnov *et al.*, 1994) as follows:

Figure No.	2	3	4
J, kg/m	10^{-9}	10^{-9}	10^{-9}
$2J/\beta$, s	10^{-2}	10^{-2}	10^{-2}
W_0, kg/(ms^2)	200	100	100
β	10^{-2}	10^{-2}	10^{-2}
f_0, Hz	2000	2000	2000
ξ, Hz	150	150	1
ε_0	10^{-3}	2.5×10^{-3}	2.5×10^{-3}

Evidently Fig. 2 represents the period doubling. But period doubling is not the only option as we mentioned earlier, and as can be seen in Fig. 3.

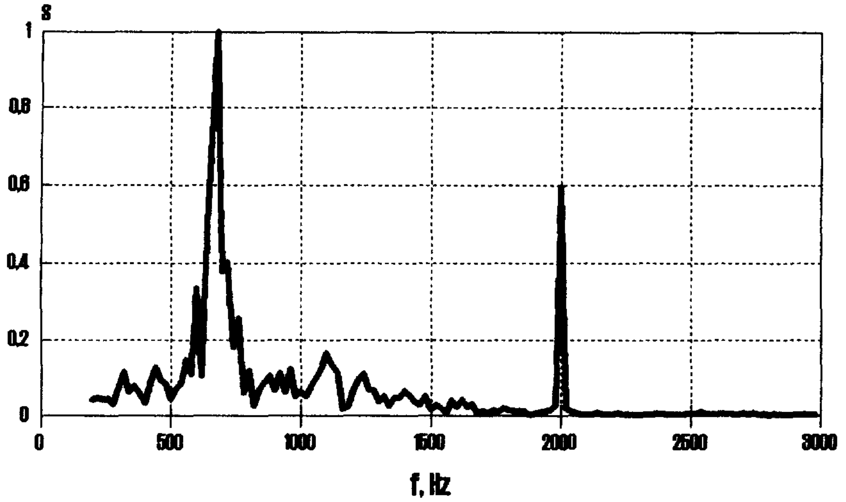

Fig. 4. Period tripling in the response spectrum in the case of a narrow frequency distribution in the initial signal.

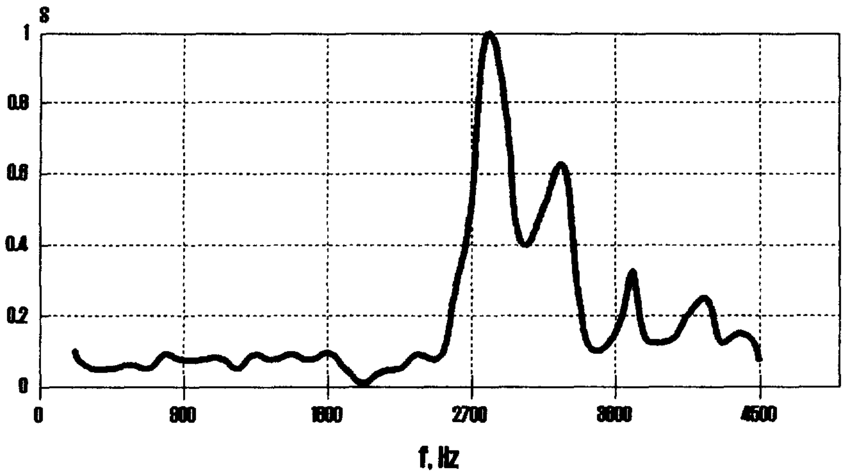

Fig. 5. Experimental initial signal spectrum measured near the source. The data was taken from Guschin *et al.* (1994).

Figure 3 corresponds to a more complicated case but the maximum amplitude at 660 Hz means period tripling. It becomes even more clear if we make the frequency distribution narrower while other values are the same, as shown in Fig. 4.

In order to compare the numerical results with experiments we show in Figs 5 and 6 two experimental curves published in (Guschin *et al.*, 1994). These curves show the spectral density of a signal propagating in soil (wet sand). The spectral density is normalized at the maximum value as before. The transformed signal in Fig. 6 looks very much like period tripling.

This example is only one of many from (Guschin *et al.*, 1994; Bubnov *et al.*, 1994) which allow us to conclude, that the numerical results given in Figs 2 and 3 qualitatively agree with the experiments. We see that a nonlinear medium with microstructure demonstrates new theoretical phenomena which are in good agreement with observations for granular materials. This provides a stimulus for further theoretical and experimental investigations in this direction.

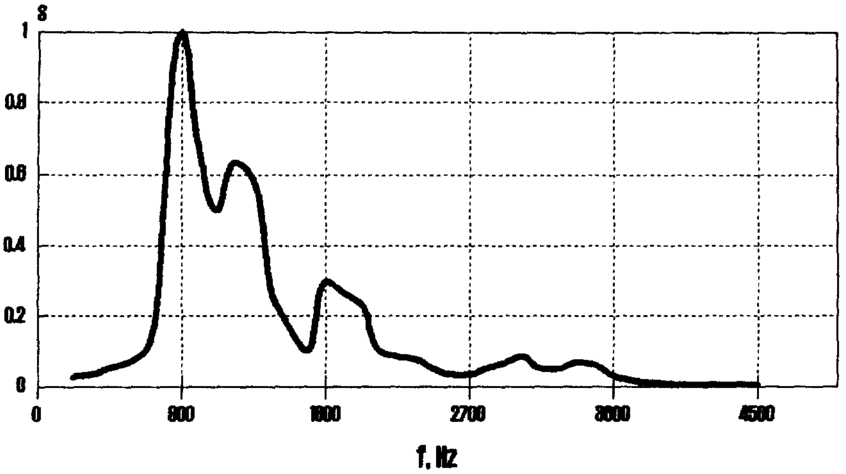

Fig. 6. Experimental signal spectrum measured at some distance from the source. The data was taken from Guschin *et al.* (1994).

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